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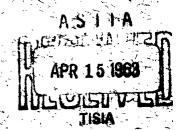
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Interim Research Memorandum

OPERATIONS EVALUATION GROUP

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INTERIM RESEARCH MEMORANDUM

OPERATIONS EVALUATION GROUP

ALLOCATION OF WEAPONS TO TARGETS WITH
EXPONENTIAL ARRIVAL TIMES IN A LIMITED
TIME INTERVAL

By J. Bram

IRM-32

ABSTRACT

Given a supply of N weapons and a time T in which to operate,
what is the best way to allocate the weapons among incoming targets
arriving with average rate λ so as to maximize the expected number
of targets killed? This problem leads to a system of ordinary dif-
ferential equations which can be solved recursively, and whose solutions
furnish the values of the expected number of targets killed, as well
as the optimal firing schedule.

Suppose that we are given N missiles which we can fire singly or in salvos at incoming targets which arrive at random times with average rate λ. We are given also an upper limit T on the amount of time we can operate. Each missile has the probability p of killing a target. What is the best way to allocate our missiles so as to maximize the expected number of targets killed?

Instead of thinking in terms of fixed N and T, let us use arbitrary n and t, and define

$$f_n(t) = \text{expected number of targets killed in the time}$$
$$\text{interval } [0, t] \text{ if we have n missiles at time}$$
$$t = 0, \text{ and we function optimally.}$$

(Of course, $f_n(t)$ depends also on p and λ.) It is clear that $f_n(0) = 0$, and $f_n(t)$ is an increasing function of n and of t. Furthermore, if $0 < t_1 < t$, then our optimal policy in the interval $(0, t)$, given n missiles at the beginning, must also be optimal in the interval (t_1, t), given $n - n_1$ missiles at t_1, if n_1 missiles were fired in $(0, t_1)$. Here, n_1 is a random variable depending on how many targets arrived in $(0, t_1)$.

Now let $t_1 = \Delta t$ be small. Then, neglecting terms in Δt^2 and higher order terms, there will be no target arrivals in $(0, \Delta t)$ with probability $1 - \lambda \Delta t$, and exactly one arrival with probability $\lambda \Delta t$. If we fire j missiles at the one target, our expected return from this salvo will be $1 - q^j$, where $q = 1 - p$. Our expected return in $(0, t)$ is therefore

$$(1 - \lambda \Delta t) f_n(t - \Delta t) + \lambda \Delta t[1 - q^j + f_{n-j}(t - \Delta t)].$$

Evidently j must be such that this expression is as large as possible. Furthermore, it can never pay to decline to fire any missiles at a target opportunity (if a bird in hand is worth two in the bush, how much better a bird in hand must be compared to one in the bush). It follows that $j \geq 1$, and

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$$f_n(t) = (1 - \lambda \Delta t) \, f_n(t - \Delta t)$$

$$+ \; \lambda \Delta t \; \max_{1 \le j \le n} \; [1 - q^j + f_{n-j}(t - \Delta t)]$$

Consequently,

$$\frac{fn(t) - f_n(t - \Delta t)}{\Delta t} \;=\; -\lambda \, f_n(t - \Delta t)$$

$$+ \; \lambda \; \max_{1 \le j \le n} \; [1 - q^j + f_{n-j}(t - \Delta t)], \tag{1}$$

and if we let $\Delta t \to 0$, we obtain, formally,

$$f_n'(t) + \lambda \, f_n(t) = \lambda \; \max_{1 \le j \le n} \; [1 - q^j + f_{n-j}(t)],$$

or

$$\frac{d}{dt} \, (e^{\lambda t} f_n(t)) = \lambda \, e^{\lambda t} \; \max_{1 \le j \le n} \; [1 - q^j + f_{n-j}(t)]. \tag{2}$$

(We can prove (2) rigorously as follows. $f_n(t)$ is a monotone function of t for each n. Therefore it is continuous almost everywhere and differentiable almost everywhere. Consequently (2) holds almost everywhere. Since $f_n(t) \le n$, the right side of (1) is a bounded function of t, and since (1) holds for all t--except for terms of order Δt--it follows that $f_n(t)$ is absolutely continuous, etc.)

From (2), we can calculate $f_n(t)$ recursively. In fact, we have evidently

$$f_o(t) = 0,$$

for all t. Therefore, from (2),

$$\frac{d}{dt}\ (e^{\lambda t}\ f_1(t)) = \lambda\ e^{\lambda t}(1 - q) = p\ \lambda\ e^{\lambda t},$$

or

$$f_1(t) = p(1 - e^{-\lambda t}). \tag{3}$$

For n = 2, we have

$$\frac{d}{dt}\ (e^{\lambda t}\ f_2(t)) = \lambda\ e^{\lambda t}\ \max
\begin{cases}
1 - q + p(1 - e^{-\lambda t}), \\
1 - q^2
\end{cases}.$$

But the maximum on the right side is the function

$$= 1 - q^2 \qquad \text{if}\quad t < \frac{1}{\lambda}\ \log \frac{1}{p}\ ,$$

$$= p(2 - e^{-\lambda t}) \quad \text{if}\quad t > \frac{1}{\lambda}\ \log \frac{1}{p}\ .$$

Consequently

$$f_2(t) = (1 - q^2)(1 - e^{-\lambda t}) \qquad \text{if}\quad t < \frac{1}{\lambda}\ \log \frac{1}{p}\ ,$$

$$= 2p - e^{-\lambda t}[3p - p^2 + \lambda pt - p\ \log \frac{1}{p}] \tag{4}$$

$$\text{if}\quad t \geq \frac{1}{\lambda}\ \log \frac{1}{p}\ .$$

In addition, we see that for n = 2, we must fire two missiles at the first target if the time remaining after its arrival is less than $\frac{1}{\lambda} \log \frac{1}{p}$, and we fire one missile at the first target in the contrary case, i.e., if the time remaining after the first arrival exceeds $\frac{1}{\lambda} \log \frac{1}{p}$.

In general, when $f_1(t)$, ..., $f_{n-1}(t)$ are known for all t, we can calculate the right side of (2) for all t, integrate, and get $f_n(t)$. This is not easy to do analytically (already for n = 3, a transcendental equation is encountered), but it is perfectly feasible on a computer. (The parameter λ does not add any complexity, of course. By changing the unit of time, we can suppose that $\lambda = 1$.)

Once we have $f_n(t)$ for $0 \leq t \leq T$, and $1 \leq n \leq N$, we also have the required optimal firing schedule. In fact, let $a_j^{(n)}$ be the value of t for which

$$1 - q^{n-j+1} + f_{j-1}(t) = 1 - q^{n-j} + f_j(t),$$

j = 1, 2, ..., n - 1. Then $0 < a_1^{(n)} < a_2^{(n)} < ... < a_{n-1}^{(n)}$; and if we have n missiles, then on the arrival of the first target, we must fire j missiles if the time remaining after this arrival is in the time interval

$$(a_{n-j}^{(n)}, a_{n-j+1}^{(n)}),$$

in which we interpret $a_0^{(n)} = 0$, $a_n^{(n)} = +\infty$.

This follows, of course, from the discussion preceding (1). After we have fired the j missiles, we have n - j left, and we employ the same philosophy when the second target arrives (given n - j missiles, this time), etc.

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It follows from the above, and it is also intuitively clear, that if we have a long time to operate, we should fire one missile at the first target, one at the second, etc. Our return will be np, if we have n missiles to start with. Therefore,

$$\lim_{t \to \infty} f_n(t) = np \tag{5}$$

(cf. (3) and (4)).

Suppose p = 1. Then the optimal policy is to fire one missile at each target. If, in the interval (0, t), there are exactly ν target arrivals, our return will be

$$= \nu, \quad \text{if} \quad \nu \leq n,$$

$$= n \quad \text{if} \quad \nu > n.$$

The probability of exactly ν arrivals in (0, t) is $e^{-\lambda t} \frac{(\lambda t)^\nu}{\nu!}$, $\nu = 0$, 1, 2, ... Therefore we have, for p = 1,

$$f_n(t) = \sum_{\nu=0}^{n} \nu \, e^{-\lambda t} \frac{(\lambda t)^\nu}{\nu!}$$

$$+ \sum_{\nu=n+1}^{\infty} n e^{-\lambda t} \frac{(\lambda t)^\nu}{\nu!} \, ,$$

or

$$f_n(t) = n + e^{-\lambda t} \left\{ (\lambda t - n) \sum_{\nu=0}^{n} \frac{(\lambda t)^\nu}{\nu!} - \frac{(\lambda t)^{n+1}}{n!} \right\} \, ,$$

(p = 1).

The most important part of our problem is the determination of the "cross-over" times $a_j^{(n)}$ defined above, in terms of which the optimal firing policy is given. For fixed n, the $a_j^{(n)}$ are bunched up near $t = 0$ for the first few j's. For small t, we have

$$f_n(t) = (1 - q^n)(1 - e^{-\lambda t}), \tag{6}$$

and the smaller n is, the larger will be the interval $(0, a_1^{(n)})$ in which (6) is valid. Now $a_1^{(n)}$ is the value of t for which

$$1 - q^n = 1 - q^{n-1} + f_1(t) = 1 - q^{n-1} + p(1 - e^{-\lambda t}).$$

Therefore

$$a_1^{(n)} = \frac{-1}{\lambda} \log (1 - q^{n-1}) . \tag{7}$$

Similarly, $a_j^{(n)}$ is the value of t for which

$$1 - q^{n-j+1} + f_{j-1}(t) = 1 - q^{n-j} + f_j(t) ,$$

which becomes, using (6)—valid for small n and small t—

$$q^{n-j} p = q^{j-1} p(1 - e^{-\lambda t}) ,$$

or

$$a_j^{(n)} = - \frac{1}{\lambda} \log (1 - q^{n-2j+1}) , \; j << n. \tag{8}$$

For large n, with $j << n$, this is practically

$$a_j^{(n)} = \frac{1}{\lambda} \; q^{n-2j+1} \; .$$

For j nearly equal to n, the most interesting case, we have not been able to derive approximate values.